Scaling laws for noise-induced temporal riddling in chaotic systems

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Recent work has considered the situation of riddling where, when a chaotic attractor lying in an invariant subspace is *transversely stable*, the basin of the attractor can be riddled with holes that belong to the basin of another attractor. The existence of invariant subspace often relies on certain symmetry of the system, which is, however, a nongeneric property as system defects and small random noise can destroy the symmetry. This paper addresses the influence of noise on riddling. We show that riddling can actually be induced by arbitrarily small noise even in parameter regimes where one expects no riddling in the absence of noise. Specifically, we argue that when there are attractors located off the invariant subspace, the basins of these attractors can be temporally riddled even when the chaotic attractor in the invariant subspace is *transversely unstable*. We investigate universal scaling laws for noise-induced temporal riddling. Our results imply that the phenomenon of riddling is robust, and it can be more prevalent than expected before, as noise is practically inevitable in physical systems. [S1063-651X(97)01410-4]

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I. INTRODUCTION

Symmetry and invariance are common in mathematical models of physical systems [1]. Such intrinsic properties in the system's equations can lead to intriguing and interesting dynamical consequences. The notion of symmetry and invariance is, however, *nongeneric* because realistic system imperfections and/or environmental noise can destroy system symmetry. An important question in the study of dynamical systems is then how the phenomena derived from the invariant properties in the models manifest themselves in practical situations where system defects and noise are inevitable.

This paper concerns a study of the effect of noise on chaotic systems with an invariant subspace. Specifically, we consider the following class of N-dimensional dynamical systems:

$$\begin{aligned} \mathbf{x}_{n+1} &= \mathbf{f}(\mathbf{x}_n) + \text{high order terms of } \mathbf{y}_n + \epsilon\sigma_n^x, \\ \mathbf{y}_{n+1} &= \mathbf{G}(\mathbf{x}_n, \mathbf{y}_n, p) + \epsilon\sigma_n^y, \end{aligned} \tag{1}$$

where $\mathbf{x}\in\mathbf{R}^{N_S}$ $(N_S\geqslant 1)$, $\mathbf{y}\in\mathbf{R}^{N_T}$ $(N_T\geqslant 1)$, $N_S+N_T=N$, $\mathbf{f}(\mathbf{x})$ is a nonlinear map that can exhibit chaos, ϵ is the noise amplitude (small), σ_n^x and σ_n^y are random variables with smooth probability distributions, and p is the bifurcation parameter. The vector function $\mathbf{G}(\mathbf{x},\mathbf{y},p)$ satisfies the condition $\mathbf{G}(\mathbf{x},\mathbf{0},p)=\mathbf{0}$ so that $\mathbf{y}=\mathbf{0}$ is an invariant subspace of the system in the absence of noise. That is, in the noiseless situation, if a trajectory starts from an initial condition with $\mathbf{y}_0=\mathbf{0}$, then the trajectory has $\mathbf{y}_n=\mathbf{0}$ for all subsequent iterations. In general, the invariant subspace can be regarded as resulting from a simple type of symmetry in the function $\mathbf{G}(\mathbf{x},\mathbf{y},p)$, e.g., $\mathbf{G}(\mathbf{x},-\mathbf{y},p)=-\mathbf{G}(\mathbf{x},\mathbf{y},p)$. In this paper we restrict our study to the case where the dynamics in the invariant subspace, governed by the map $\mathbf{f}(\mathbf{x})$, is chaotic. The largest Lyapunov exponent for the dynamics in the invariant subspace is therefore positive. For trajectories lying in the invariant subspace, their transverse stability is determined by the following largest transverse Lyapunov exponent:

$$\Lambda_T = \lim_{M\to\infty}\frac{1}{M}\sum_{n=1}^{M}\ln\left|\left.\frac{\partial\mathbf{y}_{n+1}}{\partial\mathbf{y}_n}\right|_{\mathbf{y}_n=0}\cdot\mathbf{u}\right|, \tag{2}$$

where $\mathbf{u}$ is a random unit vector in the transverse subspace $\mathbf{R}^{N_T}$.

Chaotic systems with an invariant subspace, mathematically described by Eq. (1), are of physical interest. They occur naturally in systems with spatial symmetry. Another important class of such systems is coupled oscillators that model a large variery of phenomena in physics, chemistry, biology, and ecology. Consider, for example, the following system of N coupled identical chaotic oscillators:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) + \epsilon\sum_j \mathbf{H}(\mathbf{x}_i-\mathbf{x}_j), \quad i=1,\ldots,N \tag{3}$$

where the state $\mathbf{x}_i$ of each oscillator, when isolated, is chaotic, and the coupling is represented by the strength ϵ and the function $\mathbf{H}(\mathbf{x}_i-\mathbf{x}_j)$ which satisfies the property $\mathbf{H}(\mathbf{0})=\mathbf{0}$. The synchronous state $\mathbf{x}_i(t)=\mathbf{x}_j(t)$ $(i,j=1,\ldots,N)$ is obviously a solution to Eq. (3). In such a case, the dynamical equation is identical for each oscillator so that oscillators started being synchronized remain so forever. Thus the subspace defined by $\mathbf{x}_i(t)=\mathbf{x}_j(t)$ $(i,j=1,\ldots,N)$ is an invariant subspace in which the dynamics is chaotic [2–4].

A series of recent works has revealed that there are unusual but interesting consequences in dynamical systems with an invariant subspace. A striking phenomenon is the occurrence of *riddling* in these systems [3–9]. When there is a chaotic attractor in the invariant subspace and another attractor (say, nonchaotic) off the invariant subspace, if Λ_T is negative so that the chaotic attractor is stable with respect to transverse perturbations, the basin of the chaotic attractor can be riddled with holes of arbitrarily small size belonging to the basin of the attractor off the invariant subspace [5]. An important implication of riddling is that the prediction of the final asymptotic attractor for specific parameter and initial conditions becomes extremely difficult. It has been demonstrated that the onset of riddling, or the riddling bifurcation, is typically induced by the loss of the transverse stability of some periodic orbits of low periods embedded in the chaotic attractor [8]. The riddling bifurcation is also called the bubbling bifurcation [3] from the viewpoint of invariant measures. Note that in the parameter regime where there is riddling, Λ_T remains negative. As the system parameter p changes further passing through a critical point p_c, Λ_T can become positive. This is referred to as the blowout bifurcation after which typical trajectories on the chaotic attractor become transversely unstable [7,3]. After the blowout bifurcation, riddling of the chaotic attractor in the invariant subspace disappears.

As we have mentioned, the notion of invariant subspace is *nongeneric*. An arbitrarily small amount of asymmetry in the system and/or random noise in the environment render impossible confinement of trajectories in the subspace which is invariant in the absence of asymmetry and/or noise. In realistic noisy systems, one thus expects the chaotic attractor in the invariant subspace to become a chaotic transient. An interesting question is whether the dynamical phenomenon of riddling associated with the notion of invariant subspace *persists* in practical situations where noise is inevitable. In this regard, for parameter values slightly above the riddling bifurcation (or bubbling bifurcation), it has been shown that asymmetry and/or noise convert the chaotic attractor in the invariant subspace into a superpersistent chaotic transient, and there are universal scaling laws associated with the chaotic transients [8,10]. Thus, if one observes the system in practical time scales, the superlong transient chaos can be regarded as ''sustained'' chaos. In this case, the concept of riddling is still meaningful.

In this paper we address the influence of noise on riddling in parameter regimes about the blowout bifurcation. We present analysis and numerical verification which indicate that small noise can actually *induce* riddling for the class of system described by Eq. (1) near the blowout bifurcation. Thus, quite contrary to intuition, there are situations where noise does not destroy riddling completely but instead creates riddling down to the scale comparable to the noise amplitude. Specifically, we argue that *both below and above* the blowout bifurcation, if there are coexisting attractors located off the invariant subspace, riddling in the temporal basins of these attractors can still occur when there is small-amplitude noise present. We call this type of riddling the *noise-induced temporal riddling*. For parameter values below the blowout bifurcation point, noise destroys riddling of the chaotic attractor in the invariant subspace, but noise creates temporal riddling for attractors off the invariant subspace. For parameter values above the blowout bifurcation, noise can also induce temporal riddling for the attractors off the invariant subspace. This is particularly intriguing because in this case, if there is no noise, riddling of the chaotic attractor in the invariant subspace disappears since it is transversely unstable. *The main implication is that the phenomenon of riddling may be more prevalent than expected before, as noise is inevitable in practical situations*. Thus, although invariant subspaces are atypical, physical consequences due to them persist or can even be induced by internal and/or external disturbances in realistic environments. A short account of this work has been reported in [9].

We stress that riddling induced by noise is *temporal* in the sense that the fine structure of the basin of an attractor is time varying due to noise. Given an initial condition, noise may make it eventually asymptote to one of the coexisting attractors. For the same initial condition, noise can kick it into the basin of another attractor at a different time. The riddled basins induced by noise are thus not spatially fixed in the phase space. Noise-induced riddling is therefore only temporal and its gross structure is defined only in spatial scales which are larger than the noise amplitude. This is fundamentally different from the notion of riddled basins in the absence of noise in which arbitrarily fine structures of the basin are fixed in time. In this case, riddling is spatial because no reference to time is needed.

The rest of the paper is organized as follows. In Sec. II we give a general argument to the phenomenon of noise-induced temporal riddling. In Sec. III we consider a simple model where noise-induced temporal riddling and the associated scaling laws can be obtained analytically. Based on the analysis with this solvable model, we argue that the scaling laws are universal. In Sec. IV we study a numerical model which is a realization of the general model Eq. (1) and verify the universal scaling laws. Discussions are presented in Sec. V.

II. NOISE-INDUCED TEMPORAL RIDDLING

We now present a qualitative argument for the phenomenon of noise-induced temporal riddling. Since p_c is the blowout bifurcation point, we have $\Lambda_T<0$ for $p<p_c$ and $\Lambda_T\geqslant 0$ for $p\geqslant p_c$ [11]. Let **S** be the invariant subspace. Assume there are two attractors, denoted by A and B, one above and another below **S**. When noise is absent, for $p \lesssim p_c$ there are two Cantor-like closed sets of positive Lebesgue measure in the phase space, one above and another below **S**. These sets are transversely stable. Points in the sets are attracted towards **S** and hence they belong to the basin of the chaotic attractor in **S**. Since the Cantor-like sets are closed and have positive measures, the basin of the chaotic attractor in **S** is riddled. The complement sets of the two Cantor-like closed sets are two open dense sets that belong to the basins of the attractors A and B, respectively. This situation is shown schematically in Fig. 1. For $p \gtrsim p_c$, the Cantor-like sets are still transversely stable but they have Lebesgue measure zero and hence **S** is now transversely unstable. In this case, the chaotic attractor in **S** becomes a repeller in the transverse direction, and trajectories above (below) **S** are repelled away from **S** and are eventually attracted to $A(B)$. The entire phase-space regions above and below **S** are the basins of attraction for typical trajectories to the attractors A and B, respectively. Therefore there is no riddling for $p \gtrsim p_c$ when there is no noise.

To see how small noise can induce temporal riddling, we note that the Cantor-like sets become ''fattened'' in the phase space in the presence of small noise. For both p below p_c and p above p_c, trajectories can come arbitrarily close to **S** due to the transversely stable Cantor-like sets. So, there is a nonzero probability that trajectories above **S** can be kicked across **S** and be attracted towards B due to noise, as shown in Fig. 1. The initial conditions in the fattened Cantor-like set above **S** can thus be in the basin of B (the noise-induced

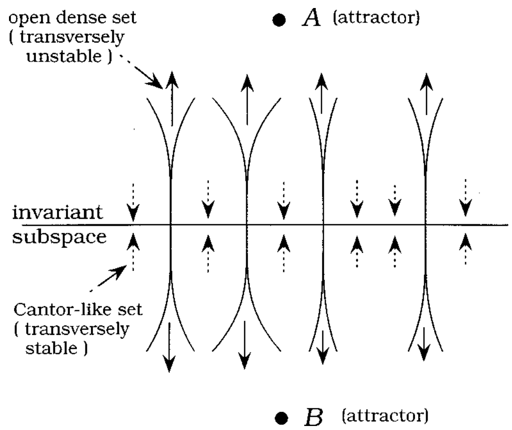

FIG. 1. A schematic illustration of two invariant sets in the phase space for p near the blowout-bifurcation point p_c. One is open dense and transversely unstable, another is transversely stable but closed. The two symmetric closed sets above and below **S** correspond to the noise-induced basins.

temporal basin of B) and form a riddled structure. By symmetry, the basin of A below **S** is also temporally riddled. We emphasize that the temporal riddling of the attractors off **S** occurs on both sides of p_c, but in the phase space, the riddling is observable only at spatial scales larger than the noise amplitude.

The above argument for noise-induced temporal riddling is only qualitative. To quantify it, we consider a simple analyzable model for which noise-induced temporal riddling can be understood fairly completely and universal scaling laws can be derived.

III. AN ANALYZABLE MODEL

A. Model description and scaling laws

We consider an analyzable model [6,9] with additive noise. For simplicity, we emphasize two-dimensional phase space, so the invariant subspace is one dimensional. The main part of the model is the following two-dimensional map defined in the region $0 \leqslant x \leqslant 1$ and $-1 < y < 1$:

$$x_{n+1} = \begin{cases} (1/a)x_n + \epsilon\sigma_n & \text{for } x_n < a \\ (1/b)(x_n - a) + \epsilon\sigma_n & \text{for } x_n > a, \end{cases} \tag{4}$$

$$y_{n+1} = \begin{cases} c y_n + \epsilon\sigma_n & \text{for } x_n < a \text{ and } |y| < 1 \\ d y_n + \epsilon\sigma_n & \text{for } x_n > a \text{ and } |y| < 1, \end{cases}$$

where $0 < a < 1$, $b = 1 - a$, $c > 1$, $0 < d < 1$, and $\epsilon\sigma_n$ is the small noise term. When $\epsilon = 0$, the invariant subspace (line) is $y = 0$ in which there is a chaotic attractor with Lyapunov exponent $\Lambda_x = a \ln(1/a) + b \ln(1/b) > 0$. The y dynamics involves both expansion and contraction in $-1 \leqslant y \leqslant 1$. Assuming that there are two fixed-point attractors located at $(\overline{x}, \pm\overline{y})$, where $\overline{y} > 1$ and $0 < \overline{x} < 1$, and further assuming that a trajectory with $|y| > 1$ approaches exponentially to one of the fixed points, we write the dynamics in the $|y| > 1$ region as

$$x_{n+1} = \overline{x} + e^{-\lambda_x}(x_n - \overline{x}), \tag{5}$$

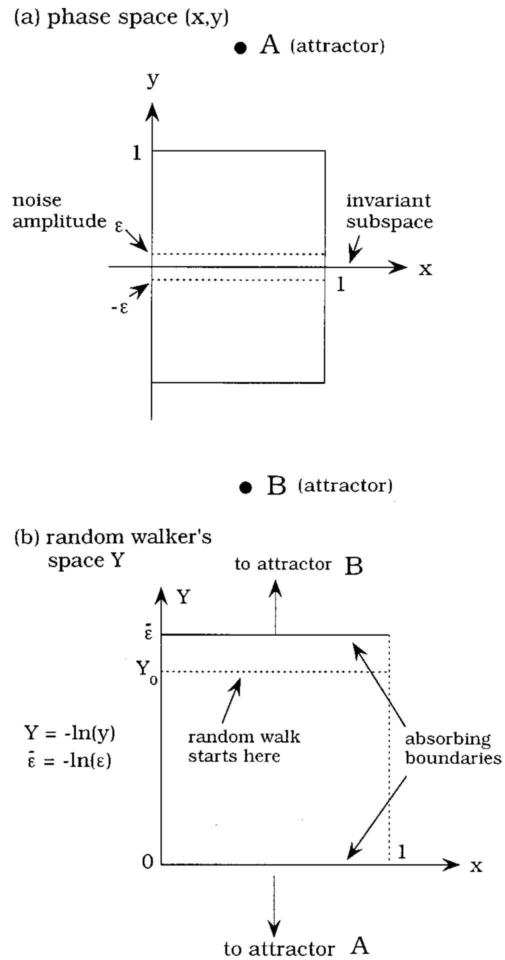

FIG. 2. Schematic illustration for (a) the analyzable model Eqs. (4) and (5), and (b) the random-walk picture with initial and boundary conditions in the diffusion approximation.

$$y_{n+1} = \begin{cases} \overline{y} + e^{-\lambda_y}(y_n - \overline{y}) & \text{for } y_n \geqslant 1 \\ -\overline{y} + e^{-\lambda_y}(y_n + \overline{y}) & \text{for } y_n \leqslant -1, \end{cases}$$

where $\lambda_x > 0$ and $\lambda_y > 0$. The model is schematically shown in Fig. 2(a). The transverse Lyapunov exponent for trajectories confined to the invariant line is

$$\Lambda_\perp = a \ln c + b \ln d.$$

Thus, if a is the bifurcation parameter, a blowout bifurcation occurs at

$$a_c = \frac{|\ln d|}{(\ln c + |\ln d|)}, \tag{6}$$

where $\Lambda_\perp \geqslant 0$ for $a \geqslant a_c$ and $\Lambda_\perp < 0$ for $a < a_c$. For $\epsilon = 0$ and $a < a_c$, the basin of the $y = 0$ chaotic attractor is riddled in the region $(0 \leqslant x \leqslant 1, -1 \leqslant y \leqslant 1)$. For $\epsilon = 0$ and $a \geqslant a_c$, this chaotic attractor is transversely unstable and, so, except for a set of Lebesgue measure zero, the upper half plane $(y > 0)$ and the lower half plane $(y < 0)$ are the basins of the $y = +\overline{y}$ and $y = -\overline{y}$ attractors, respectively. For $\epsilon \neq 0$ and a near a_c, as described in Sec. II, there is a nonzero probability that the trajectory from an initial condition in the upper (lower) unit square asymptotes to the fixed-point attractor at

$-\overline{y}\,(\overline{y})$. Thus there are now noise-induced temporal basins in the upper and lower unit squares in parameter regimes about the blowout bifurcation.

We now quantitatively characterize the noise-induced temporal riddling. First, we consider scaling of the fraction of noise-induced temporal basins. Say we fix a line segment $0 \leqslant x \leqslant 1$ at $y = y_0 \gtrsim 0$ and choose a large number of initial conditions from it. If $\epsilon \neq 0$, a fraction of these initial conditions is in the basin of the fixed-point attractor at $-\overline{y}$. Denote this fraction by F_ϵ. Apparently, F_ϵ depends both on ϵ and on y_0. For fixed y_0, as ϵ increases, we expect F_ϵ to increase because it is more likely for a trajectory to be kicked through $y=0$ by larger noise. For fixed noise amplitude ϵ, if the line segment is further away from $y=0$, it is less likely for an initial condition to be in the $-\overline{y}$ basin. Thus we expect F_ϵ to decrease as y_0 increases at fixed ϵ. We find that for small ϵ and y_0, F_ϵ obeys the algebraic scaling law

$$F_\epsilon \sim \epsilon^\mu y_0^{-\gamma}, \tag{7}$$

where $\mu > 0$ and $\gamma > 0$ are the scaling exponents that satisfy $\mu = \gamma$.

Second, we consider the probability for two nearby initial conditions to asymptote to different attractors. Specifically, we choose two initial conditions which are δ distance apart on a line segment $0 \leqslant x \leqslant 1$ at y_0, where $\delta \ll 1$. Let $\langle P_U(\delta) \rangle$ be the probability for these two initial conditions to asymptote to different attractors. This probability is thus the uncertain probability with respect to small perturbation δ. We find that $\langle P_U(\delta) \rangle$ scales algebraically with δ,

$$\langle P_U(\delta) \rangle \sim \delta^\alpha, \tag{8}$$

where the scaling exponent α is positive but its value is very close to zero; it is called the uncertainty exponent [12]. As we will see later, α being close to zero has significant consequence regarding the predictability of the final asymptotic attractor of the system for specific initial conditions and parameters.

The scaling laws (7) and (8) are the main quantitative results of this paper. We emphasize that the notion of basin here is only in the probabilistic sense. Under the influence of noise, apparently there are no fixed basins in the phase space. Temporal riddling occurs at scales which are larger than the noise amplitude. Thus, given an initial condition, the meaningful question to ask concerns the *probability* of whether this initial condition is in the $+\overline{y}$ basin or in the $-\overline{y}$ basin.

B. Random walk and diffusion approximation

Concentrating on the $y > 0$ half plane and defining $Y_n \equiv -\ln y_n$ for $y \leqslant 1$, in the noise-free case we obtain a random walk in terms of Y_n for the y dynamics: $Y_{n+1} = \gamma_n + Y_n$, where $\gamma_n = \overline{c} \equiv -\ln c < 0$ with probability a and $\gamma_n = \overline{d} \equiv -\ln d > 0$ with probability $b = 1-a$. We are interested in the parameter regime near the blowout bifurcation because in this case, *on average* the trajectory moves slowly in the y direction and, hence, the random walk can in general be solved by using the diffusion approximation. Let $P(Y, Y_0, n)$ be the probability distribution function for Y (given that x_0 is chosen randomly on the horizontal line segment at $y = y_0$, $0 \leqslant x_0 \leqslant 1$), we obtain the following diffusion equation for $P(Y, Y_0, n)$ [13]:

$$\frac{\partial P}{\partial n} + \nu \frac{\partial P}{\partial Y} = D \frac{\partial^2 P}{\partial Y^2}, \tag{9}$$

where $\nu = a\overline{c} + b\overline{d} = -\Lambda_\perp$ is the average drift, and $D \equiv \frac{1}{2}\langle(\delta Y - \langle \delta Y \rangle)^2\rangle = \frac{1}{2}ab(\overline{c}-\overline{d})^2$ is the diffusion coefficient (the average $\langle\ \rangle$ is with respect to initial random values of x_0). For concreteness, we treat the case where $a \gtrsim a_c$. The case where $a \lesssim a_c$ can be treated similarly. For $a \gtrsim a_c$ we have $\nu \lesssim 0$, indicating that on average, Y gradually approaches 0 (or $y \to 1$). For $y \gtrsim 1$, the trajectory rapidly asymptotes to the fixed point at $\overline{y}$ and hence there is an absorbing barrier for the random walker at $Y=0$. Thus we have the boundary condition

$$P(0, Y_0, n) = 0. \tag{10}$$

Since all initial conditions start from y_0, where $0 < y_0 < 1$ (or $Y_0 > 0$,) we have the initial condition

$$P(Y, Y_0, 0) = \delta(Y - Y_0). \tag{11}$$

To model the effect of noise, we note that once a trajectory falls within distance ϵ of $y=0$, it can tunnel through $y=0$ and asymptotes to the $y = -\overline{y}$ attractor. Roughly, there is an absorbing boundary for the random walker at $\overline{\epsilon} \equiv \ln(1/\epsilon) > 0$. As a crude approximation, we have the following boundary condition:

$$P(\overline{\epsilon}, Y_0, n) = 0. \tag{12}$$

Figure 2(b) shows schematically the boundary and initial conditions (10)–(12) for the random walker. The diffusion equation (9), together with Eqs. (10)–(12), can be solved by using the standard Laplace-transformation method [13]. Letting

$$\overline{P}(Y, Y_0, s) \equiv \int_0^\infty P(Y, Y_0, n) e^{-sn} dn$$

be the Laplace transform of $P(Y, Y_0, n)$, we obtain from Eq. (9) the following second-order ordinary differential equation for $\overline{P}(Y, Y_0, s)$:

$$D \frac{d^2\overline{P}(Y, Y_0, s)}{dY^2} + \Lambda_\perp \frac{d\overline{P}(Y, Y_0, s)}{dY} - s\overline{P}(Y, Y_0, s) = -\delta(Y - Y_0). \tag{13}$$

With the boundary conditions (10) and (12), and the condition that $\overline{P}(Y, Y_0, s)$ is continuous at Y_0, we obtain the solution

$$\overline{P}(Y, Y_0, s) = \begin{cases} A[e^{\lambda_1 Y} - e^{(\lambda_1 - \lambda_2)\overline{\epsilon} + \lambda_2 Y}], & Y \geqslant Y_0 \\ AB(e^{\lambda_1 Y} - e^{\lambda_2 Y}), & Y < Y_0 \end{cases} \tag{14}$$

where the coefficients A and B are given by

$$A=\frac{1}{D(\lambda_2-\lambda_1)}\frac{e^{-\lambda_2 Y_0}-e^{-\lambda_1 Y_0}}{e^{(\lambda_1-\lambda_2)\bar{\epsilon}}-1},$$
$$B=\frac{e^{\lambda_1 Y_0}-e^{(\lambda_1-\lambda_2)\bar{\epsilon}+\lambda_2 Y_0}}{e^{\lambda_1 Y_0}-e^{\lambda_2 Y_0}}, \tag{15}$$

and

$$\lambda_1=\tfrac{1}{2}\eta(\Delta-1),$$
$$\lambda_2=-\tfrac{1}{2}\eta(\Delta+1),$$
$$\eta=\Lambda_\perp/D>0,$$
$$\Delta=\sqrt{1+4Ds/(\Lambda_\perp^2)}. \tag{16}$$

C. Scaling of the fraction of noise-induced temporally riddled basins

We can now calculate F_ϵ, the fraction of the noise-induced temporally riddled basins. Since approximately, trajectories in the upper unit square falling within distance ϵ of $y=0$ are considered as being able to penetrate through $y=0$ and to asymptote to the fixed-point attractor at $-\bar{y}$, F_ϵ is roughly the total probability flux through the absorbing boundary at $Y=\bar{\epsilon}$ for the random walker. From the diffusion equation (9), the instantaneous flux through $Y=\bar{\epsilon}$ (the probability through $\bar{\epsilon}$ per unit time) is

$$f_\epsilon(n)=\delta P(\bar{\epsilon},Y_0,n)|_{Y=\bar{\epsilon}}-D\left.\frac{dP(Y,Y_0,n)}{dY}\right|_{Y=\bar{\epsilon}}$$
$$=-D\left.\frac{dP(Y,Y_0,n)}{dY}\right|_{Y=\bar{\epsilon}}. \tag{17}$$

The fraction F_ϵ is given by

$$F_\epsilon=\lim_{n\to\infty}F_\epsilon(n),$$

where $F_\epsilon(n)$ is the flux through $Y=\bar{\epsilon}$ in time n,

$$F_\epsilon(n)=\int_0^n f_\epsilon(n)dn=-\int_0^n D\left.\frac{dP(Y,Y_0,n)}{dY}\right|_{Y=\bar{\epsilon}}dn. \tag{18}$$

The Laplace transform of $F_\epsilon(n)$ is

$$\bar{F}_\epsilon(s)=\int_0^\infty F_\epsilon(n)e^{-sn}dn=-\frac{D}{s}\left.\frac{d\bar{P}(Y,Y_0,s)}{dY}\right|_{Y=\bar{\epsilon}}$$
$$=\frac{e^{\lambda_1\bar{\epsilon}}}{s}\frac{(e^{-\lambda_2 Y_0}-e^{-\lambda_1 Y_0})}{e^{(\lambda_1-\lambda_2)\bar{\epsilon}}-1}. \tag{19}$$

Note that $\bar{F}_\epsilon(s)$ has a pole at $s=0$ and a branch singularity at $s^*=-\Lambda_\perp^2/(4D)<0$ (the solution $s=0$ from $\lambda_1=\lambda_2$ is not a pole). Performing the inverse Laplace transform and noting that the contribution from the branch singularity at s^* gives a term which is proportional to e^{-s^*n} and thus vanishes in the limit $n\to\infty$, we see that the only contribution to F_ϵ comes from the pole at $s=0$. We obtain

$$F_\epsilon=\frac{e^{\eta Y_0}-1}{e^{\eta\bar{\epsilon}}-1}=\frac{y_0^{-\eta}-1}{\epsilon^{-\eta}-1}. \tag{20}$$

One may arrive at the same conclusion by computing the total probability flux into $Y=0$ $(y=1)$ (Appendix A). In the limits $\epsilon\to 0$ and $y_0\to 0$, we obtain the scaling law Eq. (7) with the scaling exponents given by

$$\mu=\gamma=\eta=\frac{\Lambda_\perp}{D}. \tag{21}$$

The equality of μ and γ can also be seen via a dimension argument. Since ϵ and y_0 have the same physical dimensions (distances), while F_ϵ is just a number, one must have $\mu=\gamma$ from Eq. (7). Due to symmetry, the same scaling holds for the fraction of the $y=+\bar{y}$ basin in the lower unit square $(0\leq x\leq 1,\ -1<y<0)$. Since the scaling exponent η depends only on $\Lambda_\perp=-\nu$ and D, which are the two fundamental parameters in the diffusion approximation, we expect the scaling law Eq. (7) to hold *universally* for noise-induced temporal riddling in the parameter regime where $\Lambda_\perp$ is small so that the diffusion approximation is valid, regardless of the details of the system.

The algebraic scaling law (7) between F_ϵ and the noise amplitude ϵ at fixed y_0 or between F_ϵ and y_0 at fixed ϵ is valid when ϵ or y_0 are small. Saying ϵ or y_0 is small is meaningful only in the relative sense: they are small compared with one, the size of phase-space region over which the diffusion process takes place. Thus, if the attractors are located far away from the invariant line $y=0$, say at $\pm\infty$, and if the diffusion process occurs in a large region, say from $Y=-\infty$ to ∞ (corresponding to from $y=0$ to $y=\infty$), we expect the algebraic scaling laws (7) to be generally valid. This can in fact be derived [9] by using a slightly modified model of Eq. (4) in which we assume that the expanding and contracting dynamics in y extends from $y=-\infty$ to $y=+\infty$. The symmetric attractors in this case are located at $y=\pm\infty$. See Appendix B for details.

D. The uncertainty exponent

The scaling law for the uncertain probability $\langle P_U(\delta)\rangle$ can be derived by using the results in Secs. III B and III C. A similar derivation for parameter regimes below but near the blowout bifurcation was done in Ref. [6], where noiseless situations were studied in which the chaotic attractor in the invariant subspace is transversely stable and riddling is with respect to this chaotic attractor. In our case, the parameter regime is immediately above the blowout bifurcation so that the chaotic attractor in the invariant subspace is transversely unstable, and riddling in our case is noise induced, is temporal, and is with respect to the attractors off the invariant subspace.

We consider two points x_0 and $(x_0+\delta)$ at $y_0\ll 1$, where $\delta\ll 1$ and both points are contained in the horizontal unit interval [0,1]. These two points thus constitute a small subinterval of size δ. Since the x dynamics is expanding, the subinterval will attain a size of order 1 after, say, $n(\delta)$ iterations. Assume that the expanded interval is still contained in [0,1] and it is vertically at y where $0<y<1$. Let $p_+(\epsilon)$ and $p_-(\epsilon)$ be the probabilities that, under the influence of noise, a randomly chosen initial condition in the horizontal unit interval at y asymptotes to the $+\overline{y}$ and $-\overline{y}$ attractors, respectively. We have $p_+(\epsilon)+p_-(\epsilon)=1$. For small noise amplitude, we have, from Eq. (20), $p_-(\epsilon)\approx(y^{-\eta}-1)\epsilon^\eta$. Since we are dealing with the parameter regime above the blowout bifurcation where the invariant line is transversely repelling, we have, typically, $y\gg y_0$ if $n(\delta)\gg 1$. Thus we expect most initial conditions in the horizontal unit interval at y to be in the temporal basin of the $y=+\overline{y}$ attractor. Hence we have $p_+(\epsilon)\approx 1$ and $p_-(\epsilon)\approx 0$. The probability $P_U(\delta)$ that x_0 and $(x_0+\delta)$ asymptote to different attractors is given by

$$P_U(\delta)=2p_+(\epsilon)p_-(\epsilon)\approx 2p_-(\epsilon)\approx 2(y^{-\eta}-1)\epsilon^\eta. \quad (22)$$

Since y can be anywhere between ϵ and 1, we see that $Y\equiv -\ln y$ can be in between 0 and $\overline{\epsilon}$ with probability distribution $P(Y,Y_0,n(\delta))$, where the time is restricted to $n(\delta)$. Thus the average value of $P_U(\delta)$, or the uncertain probability, is given by

$$\begin{aligned}\langle P_U(\delta)\rangle &\sim \int_0^{\overline{\epsilon}} 2(e^{\eta Y}-1)\epsilon^\eta P(Y,Y_0,n(\delta))dY\\ &=2\epsilon^\eta\int_0^{\overline{\epsilon}}(e^{\eta Y}-1)\\ &\quad\times\left(\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma}\overline{P_n}(Y,Y_0,s)e^{sn(\delta)}ds\right)dY,\end{aligned} \quad (23)$$

where $\overline{P_n}(Y,Y_0,s)$ is the Laplace transform of $P(Y,Y_0,n(\delta))$, and σ is chosen so that all singularities are to the left of the integration path in s and so that the integral in s converges. From Eq. (23), we see that the dependence of $\langle P_U(\delta)\rangle$ on δ only appears in $n(\delta)$. To express $n(\delta)$ in terms of δ, we refer to the random-walk picture. Assuming that it takes n_u steps of step size $\overline{d}>0$ and n_d steps of step size $\overline{c}<0$ for the walker to travel from Y_0 to Y, we have

$$\begin{aligned}Y&=Y_0+n_u\overline{d}+n_d\overline{c},\\ n(\delta)&=n_u+n_d.\end{aligned} \quad (24)$$

Noting that an upward and a downward step in Y corresponds to x's falling in the interval $(a,1)$ and $(0,a)$, respectively, we have the following condition for the small interval of size δ to expand horizontally to a length of approximately one:

$$\delta\left(\frac{1}{a}\right)^{n_d}\left(\frac{1}{b}\right)^{n_u}\approx 1. \quad (25)$$

Combining Eqs. (24) and (25), and using $\Lambda_x=a\ln(1/a)+b\ln(1/b)$ and $\Lambda_\perp=a\ln c+b\ln d=a|\overline{c}|-b\overline{d}$, we obtain

$$n(\delta)\approx\frac{(1-\Lambda_\perp/|\overline{c}|)\ln(1/\delta)+(Y-Y_0)(b/|\overline{c}|)\ln(b/a)}{\Lambda_x-(\Lambda_\perp/|\overline{c}|)\ln(1/a)}. \quad (26)$$

We note that Λ_x is on the order of 1. Immediately above the blowout bifurcation, we have $\Lambda_\perp\gtrsim 0$. Thus the terms involving $\Lambda_\perp$ in Eq. (26) are negligible, and we have

$$n(\delta)\approx\frac{1}{\Lambda_x}\ln\left(\frac{1}{\delta}\right)+(Y-Y_0)K, \quad (27)$$

where $K\equiv(1/\Lambda_x)(b/|\overline{c}|)\ln(b/a)$ is a constant. Substituting Eq. (27) into Eq. (23) and rearranging the integrations with respect to s and Y, we obtain

$$\langle P_U(\delta)\rangle=\int_{-i\infty+\sigma}^{i\infty+\sigma}\exp\left[\frac{s}{\Lambda_x}\ln\left(\frac{1}{\delta}\right)\right]H(s)ds, \quad (28)$$

where the function $H(s)$ is given by

$$H(s)=\frac{\epsilon^\eta e^{-sKY_0}}{i\pi}\int_0^{\overline{\epsilon}}(e^{\eta Y}-1)e^{sKY}\overline{P}(Y,Y_0,s)dY. \quad (29)$$

From Eq. (28), we see that the scaling of $\langle P_U(\delta)\rangle$ with δ is determined by terms which behave like

$$\exp\left[\frac{s^*}{\Lambda_x}\ln\left(\frac{1}{\delta}\right)\right]=\delta^{-s^*/\Lambda_x}, \quad (30)$$

where s^*'s are the singularities of the function $H(s)$. Explicit evaluation of $H(s)$ in Eq. (29) [14] indicates that $H(s)$ has no poles but it has a branch singularity at $s^*=-\Lambda_\perp^2/(4D)$. Substitution of this branch singularity into Eq. (28) yields the scaling law (8) with the uncertainty exponent given by

$$\alpha=\Lambda_\perp^2/(4D\Lambda_x). \quad (31)$$

In the parameter regime near the blowout bifurcation where $\Lambda_\perp\approx 0$, we see that the uncertainty exponent behaves like $\Lambda_\perp^2$, which is then very close to zero. Thus decreasing δ in a range which has the noise amplitude ϵ as its lower bound yields no substantial decrease in $\langle P_U(\delta)\rangle$. Regarding δ as the precision in the specification of the initial condition, we see that for temporally riddled basins induced by noise, improvement in δ above the noise level leads to no apparent improvement in our ability to predict, probabilistically, the asymptotic attractor. This is typical of riddled basins [5,6]. We mention that the uncertainty exponent for riddling of the chaotic attractor in the invariant subspace in noiseless situations has the same form [6].

IV. NUMERICAL VERIFICATION

To verify our assertion that the scaling laws (7) and (8) associated with noise-induced riddling are general near the blowout bifurcation regardless of details of the system, we consider the following two-dimensional version of Eq. (1):

$$\begin{aligned}x_{n+1}&=f(x_n)+qy_n^2+\epsilon\sigma_n^x,\\ y_{n+1}&=px_ny_n+y_n^3+\epsilon\sigma_n^y,\end{aligned} \quad (32)$$

where $\sigma_n^x,\sigma_n^y\in[-1,1]$ are uniform random numbers, $\epsilon\ll 1$ is the noise amplitude, $p>0$ and q are parameters. In Eq. (32), both the invariant subspace $(y=0)$ and the transverse subspace are one dimensional. Note that noise only affects the dynamics in the vicinity of the invariant line $y=0$, as the noise term is negligible when $|y|$ is large. For illustrative purpose, we choose $f(x)$ to be the doubling transformation $2x \bmod(1)$ that generates a chaotic attractor with a uniform invariant density $\rho(x)=1$ for $x\in[0,1]$ and a positive Lyapunov exponent $\Lambda_x=\ln 2$. The transverse Lyapunov exponent is given by

$$\Lambda_\perp=\int \ln|px|\rho(x)dx=\int_0^1 \ln|px|dx=\ln p-1. \quad (33)$$

The blowout-bifurcation point is $p_c=e=2.71828\ldots$. From the y equation in Eq. (32), we see that for $p>0$, if $|y_n|>1$, then $|y_{n+1}|>|y_n|$. Thus $y=\pm\infty$ are the two attractors symmetrically located with respect to $y=0$.

We first consider the noiseless case where $\epsilon=0$. For $p\lesssim p_c$, we have $\Lambda_\perp\lesssim 0$ so that the chaotic attractor of the doubling transformation at $y=0$ is also an attractor of the full phase space, the basin of which is riddled with holes belonging to the basins of the $y=\pm\infty$ attractors. This is shown in Figs. 3(a) and 3(b) for $p=2.6$ ($\Lambda_\perp\approx-0.0445$) and $q=0.1$, where the black dots denote the basin of the $y=0$ chaotic attractor. In Fig. 3(a), a grid of 2048×2048 initial conditions is chosen in the unit square $0<(x,y)\leqslant 1$. If the trajectory resulting from an initial condition reaches $y=1000$, the initial condition is regarded as belonging to the basin of the $y=+\infty$ attractor, and if the trajectory comes within 10^{-12} of $y=0$ for successive 1000 iterations, the initial condition is regarded as being in the basin of the $y=0$ attractor. Since all initial conditions have $y>0$ and since $\epsilon=0$, we observe that no trajectory goes to the $y=-\infty$ attractor. Figure 3(b) is an enlargement of the region ($0.4\leqslant x\leqslant 0.6$, $0<y\leqslant 0.05$) in Fig. 3(a) in which a grid of 1024×1024 initial conditions is chosen. The set of black dots (the basin of the $y=0$ attractor) exhibits typical features of riddling [5,6]: for any black dot, there are white regions arbitrarily nearby that correspond to the basin of the $y=+\infty$ attractor. For $p\gtrsim p_c$, we have $\Lambda_\perp\gtrsim 0$ so that the $y=0$ chaotic attractor is no longer transversely stable. In this case, $y=\pm\infty$ are the only global attractors of the system. Since $\epsilon=0$, there is no tunneling through $y=0$ of trajectories and, consequently, the basins of the $y=+\infty$ and $-\infty$ attractors are $y>0$ and $y<0$, respectively, and the basin boundary is the one-dimensional line $y=0$.

Next we consider the case where noise is present. For $p\lesssim p_c$, trajectories can no longer stay in the $y=0$ chaotic attractor forever due to noise. This attractor is therefore not a global attractor in the full phase space. Since the influence of noise is negligible for large $|y|$, the $y=\pm\infty$ attractors are still the global attractors of the system. In this case, the basin of the $y=-\infty$ ($+\infty$) attractor in $y>0$ ($y<0$) is temporally riddled, as shown (black dots) for $\epsilon=10^{-12}$ in Fig. 4(a) in which a grid of 2048×2048 initial conditions is chosen in the unit square, and an initial condition is regarded as belonging to the basin of $y=-\infty$ if the trajectory resulting from it falls below $y=-1000$. Blank regions correspond to the temporal basin of the $y=+\infty$ attractor which is determined by the numerical criterion $y>1000$. Figure 4(b) is an enlargement of the region ($0.4\leqslant x\leqslant 0.6$, $0<y\leqslant 0.05$) in Fig. 4(a), where the initial conditions are chosen over a grid of 1024×1024. Figures 4(a) and 4(b) exhibit riddling structures similar to those seen in Figs. 3(a) and 3(b), but the riddling here is noise induced, is thus temporal, and is with respect to the $y=-\infty$ attractor in $y>0$.

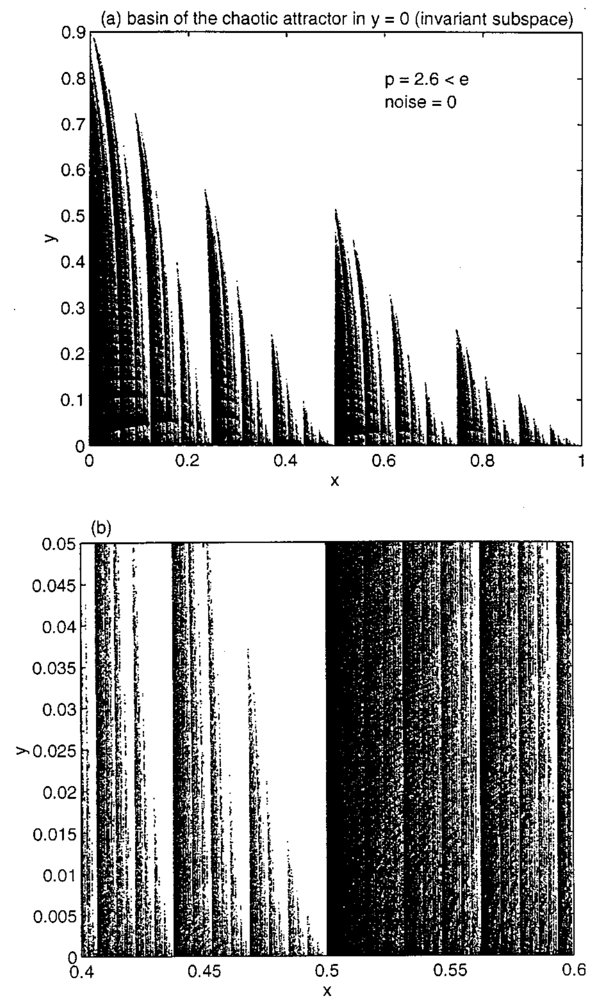

FIG. 3. When noise is absent, riddled basin (black dots) of the chaotic attractor in the invariant subspace $y=0$ below the blowout bifurcation: (a) in the unit square; and (b) enlargement of (a) in the region ($0.4\leqslant x\leqslant 0.6$, $0<y\leqslant 0.05$). Parameter setting is $p=2.6<p_c$, and $q=0.1$. The transverse Lyapunov exponent is $\Lambda_\perp\approx-0.0445$.

With noise, the interesting case is $p\gtrsim p_c$ (above the blowout bifurcation). In this case, noise can also induce temporal riddling between the basins of the $y=+\infty$ and $-\infty$ attractors, while no riddling occurs even for the $y=0$ chaotic attractor in the absence of noise. Figure 5(a) shows the basin of the $y=-\infty$ attractor in $y>0$ (black dots), where the parameter setting is $p=2.8>p_c$ ($\Lambda_\perp\approx 0.0296$), $q=0.1$, $\epsilon=10^{-12}$, and initial conditions are chosen over a grid of 4096×4096 in the unit square. Figure 5(b) shows an enlargement of part of Fig. 5(a) for a grid of 2048×2048 initial conditions in $0.4\leqslant x\leqslant 0.6$ and $0<y\leqslant 0.05$. Similar structures are observed. Note that the features exhibited in Figs. 5(a) and 5(b) are similar to those in Figs. 3(a) and 3(b), but Figs. 5(a) and 5(b) are associated with the noise-induced temporal riddling of the $y=-\infty$ attractor *above* the blowout bifurcation, whereas Figs. 3(a) and 3(b) correspond to riddling of

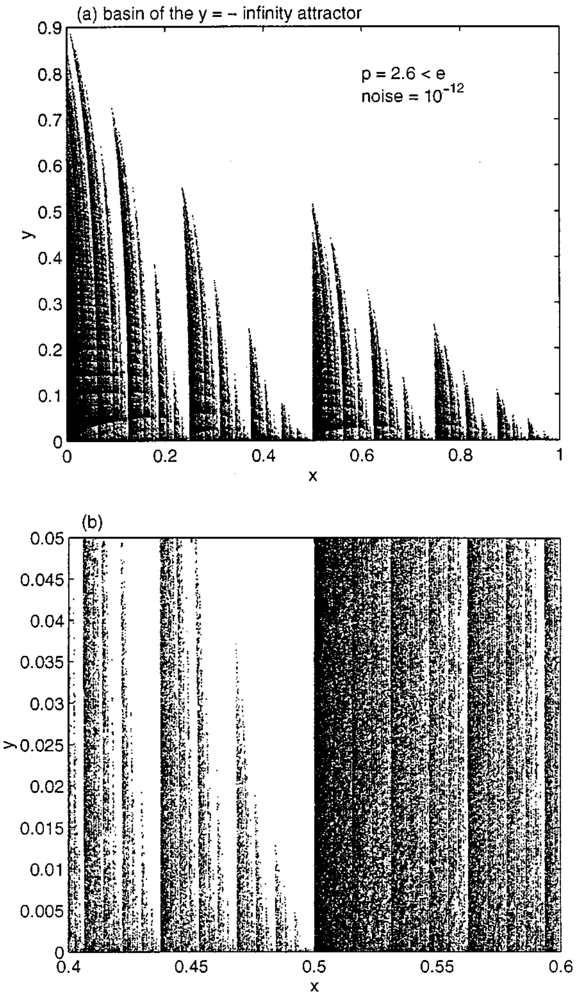

FIG. 4. For $p=2.6<p_c$ and $q=0.1$, noise-induced temporally riddled basin of the $y=-\infty$ attractor: (a) in the unit square $0<(x,y)\leqslant 1$; and (b) in the region ($0.4\leqslant x\leqslant 0.6$, $0<y\leqslant 0.05$). The noise is uniform in $[-\epsilon,\epsilon]$ with amplitude $\epsilon=10^{-12}$. Note that in this case, noise renders transversely unstable the $y=0$ chaotic attractor, but the basin of the $y=-\infty$ $(+\infty)$ attractor is temporally riddled in $y>0$ $(y<0)$ down to the scale of the noise level.

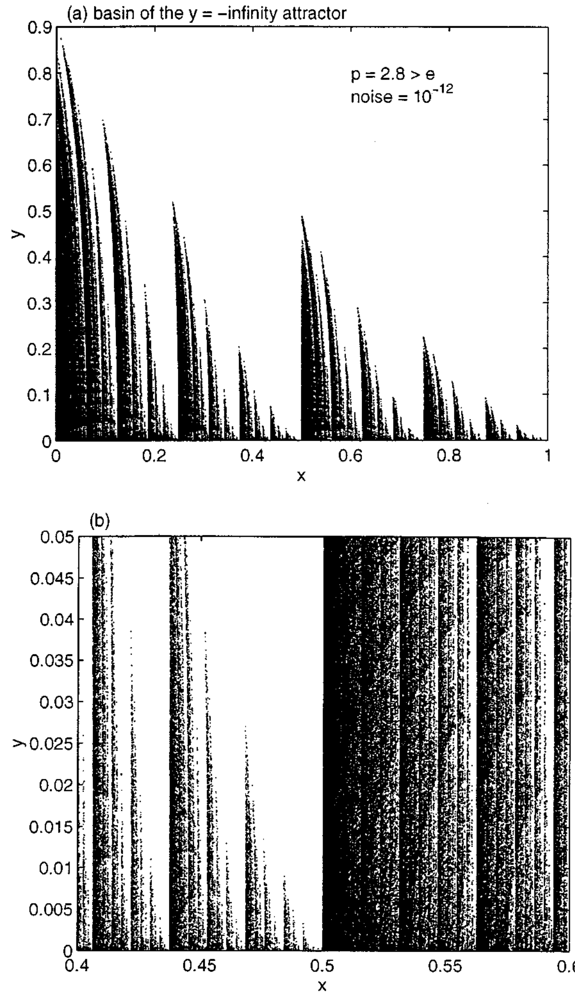

FIG. 5. For $p=2.8>p_c$ ($\Lambda_\perp\approx 0.0296$, above the blowout bifurcation) and $q=0.1$, noise-induced temporally riddled basin of the $y=-\infty$ attractor: (a) in the unit square $0<(x,y)\leqslant 1$; and (b) in the region ($0.4\leqslant x\leqslant 0.6$, $0<y\leqslant 0.05$). The noise is the same as in Fig. 4. Note that in this case, there would be no riddling if noise is absent: the entire $y>0$ half plane is the basin of the $y=+\infty$ attractor except a set of Lebesque measure zero without noise.

the chaotic attractor in the invariant subspace in the absence of noise *below* the blowout bifurcation.

To verify the scaling law (7), we first compute the fraction of points F_ϵ on a fixed line $y_0\gtrsim 0$ that belong to the temporal basin of the $y=-\infty$ attractor as ϵ changes. Figure 6 shows $\log_{10} F_\epsilon$ versus $\log_{10}\epsilon$ for $p=2.8$, $q=0.1$ and $10^{-12}<\epsilon\leqslant 10^{-6}$, where 10^6 initial conditions are chosen on the line $y=0.01$. We see that the plot can be roughly fitted by a straight line, indicating an algebraic scaling relation between F_ϵ and ϵ: $F_\epsilon\sim\epsilon^\mu$, where the scaling exponent is $\mu\approx 0.050$. Next, we compute F_ϵ at a fixed noise amplitude ϵ as y_0 $(y_0\gtrsim 0)$ increases. Figure 7 shows $\log_{10} F_\epsilon$ versus $\log_{10} y_0$ for $10^{-12}<y_0\leqslant 10^{-6}$, where $\epsilon=10^{-12}$, and 10^6 initial conditions are used. We also obtain an algebraic scaling relation: $F_\epsilon\sim y_0^{-\gamma}$, where the scaling exponent is $\gamma\approx 0.065$. We see that μ and γ have similar values.

To check the universality of the scaling law (7), we note that in our numerical model, the transverse Lyapunov exponent is $\Lambda_\perp=\ln p-1$ and the diffusion coefficient is given by

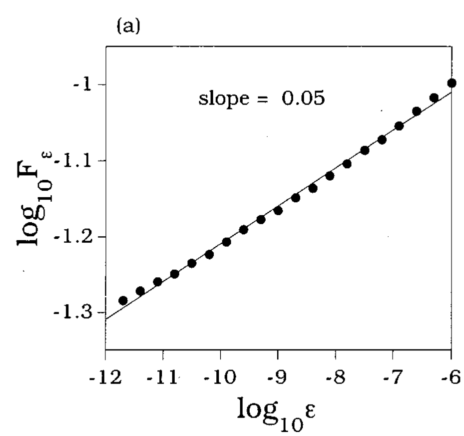

FIG. 6. At $y_0=0.01$, on a logarithmic scale, the probability F_ϵ that a random x_0 asymptotes to the $y=-\infty$ attractor versus the noise amplitude ϵ. The plot indicates that roughly, $F_\epsilon\sim\epsilon^{0.05}$. Other parameters are $p=2.8$ and $q=0.1$.

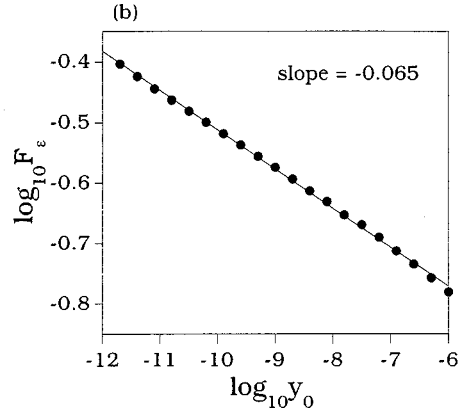

FIG. 7. At $\epsilon=10^{-12}$, for $p=2.8$ and $q=0.1$, F_ϵ versus y_0 on a logarithmic scale. Roughly, we have $F_\epsilon \sim y_0^{-0.065}$.

$$D=\frac{1}{2}\int\left(\left.\frac{\partial y_{n+1}}{\partial y_n}\right|_{y_n=0}-\left\langle\left.\frac{\partial y_{n+1}}{\partial y_n}\right|_{y_n=0}\right\rangle\right)^2\rho(x_n)dx_n$$

$$=\frac{1}{2}\int[\ln(px)-\Lambda_\perp]^2\rho(x)dx=\frac{1}{2}. \quad (34)$$

Thus we have for the algebraic scaling exponent $\eta=2(\ln p-1)$ for $p \gtrsim p_c$. For $p=2.8$, we have $\eta=0.059$. This agrees fairly well with the numerical values of $\mu\approx 0.050$ and $\gamma\approx 0.065$.

We now compute the scaling of the uncertain probability $\langle P_U(\delta)\rangle$. To do this we fix $p=2.8$ and $q=0.1$ (the same parameter setting as in Figs. 5 and 6), and choose a large number of pairs (N_0) of initial conditions of distance δ apart at $y_0=10^{-6}$ under noise of amplitude $\epsilon=10^{-15}$. This rather small noise level is used because $\langle P_U(\delta)\rangle$ is meaningful only when $\delta \gg \epsilon$. For each pair, we determine if the initial conditions go to different attractors. If yes, this pair is uncertain with respect to perturbation δ. For fixed δ, we increase N_0 until the number of uncertain pairs of initial conditions reaches 1000. We then have $\langle P_U(\delta)\rangle \approx 1000/N_0$, where $N_0 \geqslant 1000$. Figure 8 shows $\log_{10}\langle P_U(\delta)\rangle$ versus $\log_{10}\delta$ for $\epsilon \ll 10^{-10} \leqslant \delta \leqslant 10^{-2}$. We see that the plot can be roughly fitted by a straight line, indicating the algebraic scaling law (8). The slope of the straight line, or the uncertainty exponent, is 0.006 ± 0.001. The theoretical prediction, however, gives $\alpha=(\ln p-1)^2/(4D\ln 2)\approx 6\times 10^{-4}$, which is one order of magnitude smaller than the numerical value. This rather large discrepancy is, however, somewhat expected because the theoretical exponent is proportional to the square of the transverse Lyapunov exponent $\Lambda_\perp$, which is itself a very small number near the blowout bifurcation. We see from Fig. 8 that the numerical fluctuations in $\langle P_U(\delta)\rangle$ are quite large. It is thus impractical to expect to be able to extract a slope from Fig. 7 that is on the order of 10^{-4} with good precision. Increasing p further away from p_c yields larger values for $\Lambda_\perp$. Therefore one might expect to obtain an improved agreement in α when $p>p_c$ but not close to p_c. However, the theoretical prediction (8) is only valid for $p \gtrsim p_c$. Thus we are forced to rely on qualitative agreement for the scaling law between $\langle P_U(\delta)\rangle$ and δ, which appears to be algebraic from Fig. 8. Numerical computation does indicate, nevertheless, that the uncertainty exponent for noise-induced temporal riddling is extremely small, as we have verified for a number of parameter values in the vicinity of p_c. Such small values of the uncertainty exponent imply an extreme insensitivity of the uncertain probability $\langle P_U(\delta)\rangle$ to changes in the precision of the initial condition. Figure 8 indicates that $\langle P_U(\delta)\rangle$ decreases only slightly when one raises the precision (corresponding to decrease δ) over eight orders of magnitude. Therefore, in practical terms, any attempt in a hope to better predict the system's asymptotic attractor by improving the measurement of initial conditions and parameters of the system will fail for situations of noise-induced temporal riddling.

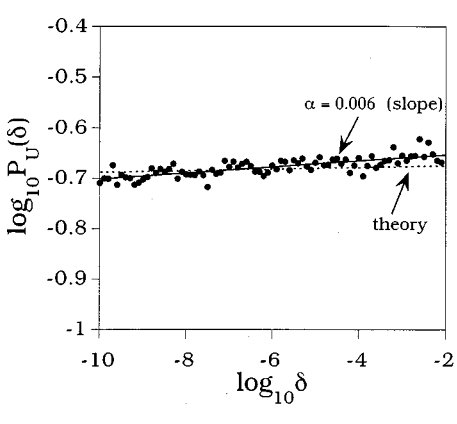

FIG. 8. The uncertainty probability $\langle P_U(\delta)\rangle$ versus δ on a logarithmic scale. The plot can be roughly fitted by a straight line, verifying the scaling law (8). The dashed line is the theoretical prediction. Note that the uncertainty exponent is very close to zero.

V. DISCUSSIONS

The presence of invariant properties is common in theoretical models of natural systems. Symmetry is perhaps one of the most often encountered properties in physical, chemical, and biological systems. Such invariant properties usually lead to interesting dynamical consequences and hence they have been tremendously helpful to our understanding of the system's dynamics. One should, however, be cautious about these invariant properties because they are usually not generic. That is to say, any defect in the system or small external noise could completely wipe out properties such as symmetry. A key question is therefore whether the physical consequences caused by the system's invariant properties still persist in noisy environment, and how.

When the system is chaotic and its equations have a simple kind of symmetry, situations often arise where there is an invariant subspace and there is a chaotic attractor in the invariant subspace. If the system is perfect and there is no noise, the presence of a chaotic attractor in the invariant subspace can lead to unusual but interesting dynamical phenomena such as riddling and on-off intermittency [16,17], which have recently become an active forefront research area in chaotic dynamics. Since the occurrence of riddling depends on the system's possessing a perfect invariant subspace, an important question is whether riddling is still observable in practical situations where noise is inevitable. In this regard, recent work has shown that in a parameter regime near the birth of riddling (riddling bifurcation), which is triggered by the loss of transverse stability of some low-period periodic orbits embedded in the chaotic attractor in the invariant subspace, riddling manifests itself [8,10] in a form of superpersistent chaotic transient [15]. The lifetime of such a chaotic transient is so extremely long when symmetry breaking and/or the noise amplitude are small, that riddling is practically observable near the riddling bifurcation. Blowout bifurcation, on the other hand, occurs when a system parameter increases further away from the riddling bifurcation point and when typical trajectories in the chaotic attractor in the invariant subspace become transversely unstable. After the blowout bifurcation, there are infinitely more periodic orbits embedded in the chaotic attractor that are transversely unstable than those that are transversely stable [18]. A question that has remained uninvestigated is whether riddling is still observable when the system is in a noisy environment and is in the parameter regime about the blowout bifurcation.

This paper gives an affirmative answer to the above question. In particular, we argue that when there are attractors (not necessarily chaotic) located off the invariant subspace, a situation easily encountered in chaotic systems, small noise can *in fact* induce temporal riddling between the basins of these attractors even beyond the blowout bifurcation. We note that near but below the blowout bifurcation, noise destroys riddling of the chaotic attractor in the invariant subspace and replaces it by a chaotic transient which is *not* superpersistent [6]. Beyond the blowout bifurcation, riddling disappears because the chaotic attractor in the invariant subspace is unstable. Thus we see that riddling of the chaotic attractor in the invariant subspace is practically *unobservable* in parameter regimes about the blowout-bifurcation point. What is physically observable in this case is the temporal riddling between attractors off the invariant subspace, which occurs both *below* and *above* the blowout-bifurcation point, as demonstrated qualitatively and quantitatively in this paper. Thus, in different forms, riddling can occur in wide parameter regimes about the blowout-bifurcation point. The universal scaling laws associated with the noise-induced temporal riddling have been obtained in this paper. Since noise is inevitable in reality, we expect riddling to occur commonly in dynamical systems with symmetry.

ACKNOWLEDGMENTS

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APPENDIX A: FLUX THROUGH $Y=0$ IN THE DIFFUSION APPROXIMATION

The instantaneous probability flux through the absorbing boundary at $Y=0$ from *above* in Y is

$$f_0(n) = -\delta P(0,Y_0,n)|_{Y=0} + D\left.\frac{dP(Y,Y_0,n)}{dY}\right|_{Y=0} = D\left.\frac{dP(Y,Y_0,n)}{dY}\right|_{Y=0}.$$

The flux through $Y=0$ from above in time n is $F_0(n) = \int_0^n f_0(n)dn$, for which the Laplace transform is

$$\overline{F_0}(s) = \frac{D}{s}\left.\frac{d\overline{P(Y,Y_0,s)}}{dY}\right|_{Y=0} = -\frac{1}{s}\frac{e^{\lambda_1 Y_0} - e^{(\lambda_1-\lambda_2)\bar{\epsilon}+\lambda_2 Y_0}}{e^{(\lambda_2+\lambda_1)Y_0}[e^{(\lambda_1-\lambda_2)\bar{\epsilon}}-1]}.$$

Performing the inverse Laplace transform and taking the limit $n\to\infty$ (so only the pole at $s=0$ has a contribution to the flux), we obtain the following fraction of initial conditions that asymptote to the fixed-point attractor at $\bar{y}$:

$$F_0 = \frac{e^{\eta\bar{\epsilon}} - e^{\eta Y_0}}{e^{\eta\bar{\epsilon}}-1}.$$

Thus we have $F_\epsilon + F_0 = 1$, which is expected because eventually all initial conditions chosen from the upper unit square asymptote either to the attractor at $\bar{y}$ or to the one at $-\bar{y}$. In fact, it is straightforward to check from direct integration that the total fraction of trajectories in $\epsilon \leqslant y \leqslant 1$ is zero in the limit of $n\to\infty$. Denote this fraction by F_+. We have $F_+ = \lim_{n\to\infty} F_+(n)$, where

$$F_+(n) = \int_\epsilon^1 P(y,y_0,n)dy.$$

The Laplace transform of $F_+(n)$ is

$$\overline{F_+}(s) = \int_0^{\bar{\epsilon}} \overline{P(Y,Y_0,s)}dY = -\frac{1}{s} + \frac{1}{s}\frac{e^{\lambda_1 Y_0}(e^{\lambda_1\bar{\epsilon}}-1) + e^{\lambda_1\bar{\epsilon}+\lambda_2 Y_0}(e^{-\lambda_2\bar{\epsilon}}-1)}{e^{(\lambda_1+\lambda_2)Y_0}[e^{(\lambda_1-\lambda_2)\bar{\epsilon}}-1]}.$$

Picking the contribution from the pole at $s=0$, we obtain $F_+=0$ in the $n\to\infty$ limit.

APPENDIX B: FRACTION OF NOISE-INDUCED TEMPORALLY RIDDLED BASINS FOR MAPS WITH ATTRACTORS AT INFINITY

We consider a slightly modified map of Eq. (4) in which the y map is no longer restricted to $|y| \leqslant 1$. In this case, the random-walk picture is good for $-\infty < Y < \infty$ (corresponding to $0 < y < +\infty$). The diffusion approximation is thus valid near the blowout bifurcation for $-\infty < Y < \infty$. In this case, the absorbing boundary condition at $Y=0$ ($y=1$) does not apply. With the initial condition Eq. (11) and the boundary condition Eq. (12), we obtain the solution to the diffusion equation,

$$\overline{P}(Y,Y_0,s)=\begin{cases} C_1 e^{\lambda_1 Y}+C_2 e^{\lambda_2 Y} & \text{for } Y>Y_0 \\ C_3 e^{\lambda_1 Y} & \text{for } Y<Y_0, \end{cases} \tag{B1}$$

where the coefficients are

$$C_2=\frac{1}{D(\lambda_1-\lambda_2)}\exp(-\lambda_2 Y_0),$$

$$C_1=-C_2\exp[(\lambda_2-\lambda_1)\overline{\epsilon}], \tag{B2}$$

$$C_3=C_2\{\exp[(\lambda_2-\lambda_1)Y_0]-\exp[(\lambda_2-\lambda_1)\overline{\epsilon}]\}.$$

Let $F_+(n)$ be the probability that the walker has not reached within ϵ of $y=0$ at time n. The Laplace transform of $F_+(n)$ is given by

$$\overline{F_+}(s)=\int_{-\infty}^{\overline{\epsilon}}\overline{P}(Y,Y_0,s)dY. \tag{B3}$$

Substituting Eqs. (B1) and (B2) into Eq. (B3), we obtain

$$\overline{F_+}(s)=1/s-(1/s)\exp[-\lambda_2(Y_0-\overline{\epsilon})]. \tag{B4}$$

Performing the inverse-Laplace transform by noting that there are a pole at $s=0$ and a branch singularity at $s=s^*\equiv-\Lambda_\perp^2/4D<0$, we obtain

$$F_+(n)=1-\exp[-\lambda_2(s=0)(Y_0-\overline{\epsilon})]$$
$$-\frac{1}{s^*}\exp[-\lambda_2(s=s^*)(Y_0-\overline{\epsilon})]\exp(s^* n).$$

In the limit $n\to\infty$, $F_+(n)$ is the probability that the random walker has never reached $Y\geqslant\overline{\epsilon}$ $(y\leqslant\epsilon)$ and hence $F_+(\infty)$ is the fraction of the $y=+\infty$ basin in the upper half plane. Therefore the noise-induced fraction of points at $y_0>0$ that belong to the $y=-\infty$ basin is given by

$$F_\epsilon=1-\lim_{n\to\infty}F_+(n)=\exp[\eta(Y_0-\overline{\epsilon})]\sim\epsilon^\eta y_0^{-\eta},$$

which is the scaling law (7).

[1] M. Field and M. Golubitsky, *Symmetry in Chaos: A Search for Pattern in Mathematics, Art and Nature* (Oxford University Press, Oxford, 1992).

[2] A. Pikovsky and P. Grassberger, J. Phys. A **24**, 4587 (1991); A. Pikovsky, M. G. Rosenblum, and J. Kurths, Europhys. Lett. **34**, 165 (1996); M. Ding and W. Yang, Phys. Rev. E **54**, 2489 (1996).

[3] P. Ashwin, J. Buescu, and I. N. Stewart, Phys. Lett. A **193**, 126 (1994); Nonlinearity **9**, 703 (1996); P. Ashwin, P. J. Aston, and M. Nicol, University of Surrey, Technical Report in Mathematics and Statistics, 1996 (unpublished); P. Ashwin and E. Stone, Centre for Interdisciplinary Nonlinear Mathematics Technical Report, 1997 (unpublished); S. C. Venkataramani, B. Hunt, E. Ott, D. J. Gauthier, and J. C. Bienfang, Phys. Rev. Lett. **77**, 5361 (1996).

[4] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. Lett. **73**, 3528 (1994).

[5] J. C. Alexander, J. A. Yorke, Z. You, and I. Kan, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 795 (1992); I. Kan, Bull. Am. Math. Soc. **31**, 68 (1994); Y.-C. Lai and C. Grebogi, Phys. Rev. E **52**, R3313 (1995); H. Nakajima and Y. Ueda, Physica D **99**, 35 (1996).

[6] E. Ott, J. C. Sommerer, J. C. Alexander, I. Kan, and J. A. Yorke, Phys. Rev. Lett. **71**, 4134 (1993); E. Ott, J. C. Alexander, I. Kan, J. C. Sommerer, and J. A. Yorke, Physica D **76**, 384 (1994).

[7] E. Ott and J. C. Sommerer, Phys. Lett. A **188**, 39 (1994).

[8] Y.-C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).

[9] Y.-C. Lai and C. Grebogi, Phys. Rev. Lett. **77**, 5047 (1996).

[10] S. C. Venkataramani, B. Hunt, and E. Ott, Phys. Rev. E **54**, 1346 (1996).

[11] The model system Eq. (1) has the property that the equation of motion in the invariant subspace $\mathbf{y}=\mathbf{0}$ is independent of the bifurcation parameter. Ashwin and co-workers call such parameters *normal parameters* [3]. In reality there can be situations where this is not true. For systems whose dynamics in the invariant subspace is also influenced by the blowout-bifurcation parameter p, our consideration in this paper applies if there is a chaotic attractor in the invariant subspace. As p is changed, the attractor itself can undergo bifurcations. Thus the transverse Lyapunov exponent $\Lambda_\perp(p)$ is not necessarily a smooth or even a continuous function of p in the vicinity of the blowout bifurcation, as is true for the plots of the Lyapunov exponents of typical chaotic systems versus some parameter. However, in general, we expect $\Lambda_\perp(p)$ to have a smooth envelope. Our results in this paper should hold with respect to the smooth envelope of the function $\Lambda_\perp(p)$. On the other hand, situations of normal parameters arise naturally in synchronization of identical coupled chaotic oscillators.

[12] C. Grebogi, S. W. McDonald, E. Ott, and J. A. Yorke, Phys. Lett. **99A**, 415 (1983); S. W. McDonald, C. Grebogi, E. Ott, and J. A. Yorke, Physica D **17**, 125 (1985).

[13] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1966).

[14] Explicit integration in Eq. (29) yields

$$H(s)=(A\epsilon^\eta/i\pi)e^{-sKY_0}\{-[B(e^{r_1Y_0}-1)+(e^{r_1\overline{\epsilon}}-e^{r_1Y_0})]/r_1+[B(e^{r_2Y_0}-1)+(e^{r_2\overline{\epsilon}}-e^{r_2Y_0})]/r_2+[B(e^{r_3Y_0}-1)+e^{(\lambda_1-\lambda_2)\overline{\epsilon}}(e^{r_3\overline{\epsilon}}-e^{r_2Y_0})]/r_3-[B(e^{r_4Y_0}-1)+e^{(\lambda_1-\lambda_2)\overline{\epsilon}}(e^{r_4\overline{\epsilon}}-e^{r_4}Y_0)]/r_4\},$$

where A and B are given by Eq. (15), $r_1\equiv\lambda_1+sK$, $r_2\equiv r_1+\eta$, $r_3\equiv\lambda_2+sK$, and $r_4\equiv r_3+\eta$. Note that $r_i=0$ $(i=1,\ldots,4)$ in $H(s)$ are not poles. Therefore the only singularity in $H(s)$ is the square-root branch singularity contained in λ_1 and λ_2.

[15] Superpersistent chaotic transients also occur in a variety of systems. See, for example, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **50**, 935 (1983); Erg. Th. Dyna. Sys. **5**, 341 (1985); J. P. Crutchfield and K. Kaneko, Phys. Rev. Lett. **60**, 2715 (1988); A. Hastings and K. Higgins, Science **263**, 1133 (1994); Y.-C. Lai and R. L. Winslow, Phys. Rev. Lett. **74**, 5208 (1995); R. Braun and F. Feudel, Phys. Rev. E **53**, 6562 (1996).

[16] E. A. Spiegel, Ann. (N.Y.) Acad. Sci. **617**, 305 (1981); A. S. Pikovsky, Z. Phys. B **55**, 149 (1984); H. Fujisaka and T. Yamada, Prog. Theor. Phys. **74**, 919 (1985); **75**, 1087 (1986); A. S. Pikovsky, Phys. Lett. A **165**, 33 (1992); N. Platt, E. A. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993); Y.-C. Lai, Phys. Rev. E **53**, R4267 (1996); **54**, 321 (1996).

[17] The effect of noise on on-off intermittency has been investigated. See, for example, J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994); S. C. Venkataramani, T. M. Antonsen, Jr., E. Ott, and J. C. Sommerer, Physica D **96**, 66 (1996).

[18] Y. Nagai and Y.-C. Lai, Phys. Rev. E **55**, R1251 (1997).